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# An approximation for the highest gravity waves on water of finite depth

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Planar steady gravity waves of finite amplitude at the surface of an ideal incompressible fluid above a flat bottom are studied theoretically. A new approach to the construction of some steady flows of heavy fluid with a partially free surface is proposed. The hypothesis is suggested and justified that these flows are close to gravity waves. For the case of the highest waves a one-parameter family of exact solutions describing free boundary flows above a flat bottom and under two uneven symmetrically located caps is derived. This family of solutions gives an approximation to the highest water waves in moderate to shallow water depths, enabling relatively simple calculation of their properties.

## 1. Introduction

The problem of gravity waves at the surface of a fluid of finite depth with one trough and crest per period has been the subject of intensive studies for some time. This problem is known to have a two-parameter family of solutions. These parameters are dimensionless wavelength and dimensionless amplitude, i.e. ratios of wavelength and amplitude to the mean depth. Wavelength varies from 0 to $\infty$, and amplitude from 0 to some limiting value depending on the wavelength. As a rule, the difficulties of theoretical description of gravity waves increase with amplitude. The problem of maximal-amplitude waves is the most complicated and interesting case. The family of these waves is one-parameter, depending on a dimensionless wavelength. On the free surface of the highest waves singularities arise, which are sharp crests with the interior angle of 120°. Stokes (1880) was the first to deduce this phenomenon. This hypothesis was justified rigorously in the papers by Toland (1978), Amick, Fraenkel & Toland (1982) and Plotnikov (1983). An asymptotic expansion in the vicinity of a singular point was found by Grant (1973), Norman (1974), Amick & Fraenkel (1987) and McLeod (1987).

The first numerical study of the highest gravity waves was performed in the papers by Michell (1893) and Havelock (1919). From more recent works, we mention the articles by Schwartz (1974) and Cokelet (1977), where a study of arbitrary-amplitude waves including the steepest one was presented. Rigorous calculations of a considerable number of parameters of the highest waves were made by Williams (1981, 1985). Very careful numerical calculation of the almost highest waves was performed by Chandler & Graham (1993) and Tanaka (1995).

Apart from the numerical calculations, a number of approximate theories exist. The linear theory of small-amplitude waves and the nonlinear long waves theory are unsuitable to describe finite-amplitude waves and especially the maximal amplitude ones. The former yields sinusoidal waves and the latter gives cnoidal waves, which do not possess the singularities at the free surface. Davies's (1951) theory of finite-amplitude waves is well-known. In this theory the initial nonlinear boundary value problem is changed slightly so that the new nonlinear problem admits an exact solution. Longuet-Higgins (1973) proposed the 'hexagon' approximation for the highest deep-water wave. A simple and accurate approximation has been derived by Longuet-Higgins & Fox (1977, 1978, 1996) for the almost highest waves.

Another approximate theory of steepest waves is developed here. The difference from the earlier theories is that our approximate solutions satisfy the hydrodynamic equations exactly. These solutions describe steady flows of a heavy fluid differing from water waves with a constant free surface pressure. So the new family of exact solutions is added to the known free-boundary flow examples (see Gurevich 1965; Milne-Thomson 1967; Daboussy, Dias & Vanden-Broeck 1997).

In this paper the water wave problem is studied as a nonlinear boundary value-problem in an infinite strip of the complex potential plane. Naturally, the solution of this problem will be a function periodic along the strip for the periodic gravitational waves. We propose to seek a solution which is periodic in the crosswise direction. As is shown in § 2, this solution agrees with the theory of cnoidal waves. In § 3 we notice that the periodic solution in the crosswise direction on the strip may be found exactly if the period is a rational number. To find the solution one has to solve a special nonlinear system of ordinary differential equations. The solution periodic across the strip with a period 6 is presented in the § 4. The prime goal of the paper is to demonstrate that the solution with this period is close to the highest water waves. We check this closeness in § 5 where the solution obtained is compared with the numerical results by Williams (1981). Good agreement is observed for long and medium wavelengths. Simple formulae, which may be useful for engineering calculations, are derived for the main integral quantities, pressure and velocity along the bottom for the highest waves. In the concluding § 6 we discuss the results obtained.

## 2. Cnoidal waves

Let the free surface be stationary and periodic and the fluid be flowing from the left to the right. Let the origin of the Cartesian coordinate system be at the bottom, the $X$-axis be directed along the bottom from the left to the right and the $Y$-axis be upright. We shall consider flows symmetrical with respect to the $Y$-axis, i.e. this axis will pass either through a crest or through a trough of the wave. Let the value of stream function $\Psi$ be zero at the bottom and $\Psi = \Psi_0 > 0$ at the free surface. Designate by $g$ the gravitational acceleration, and by $u_0, h_0$ the velocity and the depth at some point of the free surface. Throughout the paper we suppose that this point is on the $Y$-axis (the point $C$ in figure 1$a$) except for § 4 where a solitary wave problem is considered. In this case the point is placed at infinity.

A fluid will occupy a strip of width 1 (figure 1$b$) in the plane of dimensionless complex potential

$$\chi = \varphi + \mathrm{i}\psi = (\Phi + \mathrm{i}\Psi)/\Psi_0.$$

It is necessary to find the function

$$Z = X + \mathrm{i}Y = h_0 f(\chi),$$

realizing the conformal mapping of this strip on the unknown flow region in the physical plane with the normalizing conditions

$$f(0) = 0, \qquad f(\mathrm{i}) = \mathrm{i}. \tag{2.1}$$

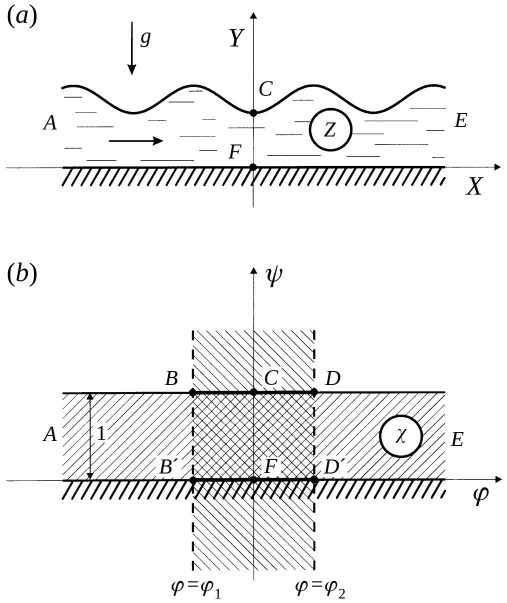


FIGURE 1. (*a*) Physical flow region. (*b*) Plane of the complex potential $\chi$.

The Bernoulli integral

$$\frac{1}{2}\left|\frac{\mathrm{d}(\Phi + \mathrm{i}\Psi)}{\mathrm{d}Z}\right|^2 + gY = \tfrac{1}{2}u_0^2 + gh_0$$

should be fulfilled at the free surface. In dimensionless variables it may be rewritten in the form

$$\left|\frac{\mathrm{d}f}{\mathrm{d}\chi}\right|^2 = \frac{\delta^2}{1 + 2(1 - \mathrm{Im}\, f)/F^2}.$$

Two dimensionless real constants arise here:

$$F = u_0/(gh_0)^{1/2}, \quad \delta = \Psi_0/(u_0 h_0).$$

If we represent the conformal mapping as

$$f(\chi) = \alpha\chi + W(\chi),$$

then, another constant $\alpha$ appears. Denoting

$$\beta = 1 + 2(1 - \alpha)/F^2,$$

we see that the water waves problem reduces to:

PROBLEM 1. *Find the constants* $\alpha, \delta, F$ *and the function* $W(\chi)$ *which are analytic in the strip*

$$0 < \psi < 1, \quad -\infty < \varphi < \infty \tag{2.2}$$

*and satisfy the following boundary conditions:*
*constant pressure condition*

$$\left|\frac{\mathrm{d}W}{\mathrm{d}\chi} + \alpha\right|^2 = \frac{\delta^2}{\beta - 2\,\mathrm{Im}\, W/F^2} \qquad (\psi = 1) \tag{2.3}$$

*flat bottom condition*

$$\mathrm{Im}\, W = 0 \qquad (\psi = 0). \tag{2.4}$$

It seems natural to seek the exact solution of this problem $W(\varphi + \mathrm{i}\psi)$ which is periodic on $\varphi$. However, the shallow-water theory shows that it would also appear reasonable to consider $W(\varphi + \mathrm{i}\psi)$ as a periodic function of variable $\psi$.

The shallow-water theory describes small-amplitude long waves. A regular procedure for finding the higher-order expansion for this theory was suggested by Keller (1948). Commonly this expansion is constructed using different dilatations along two different directions. As a result, the analyticity of a complex variable for the functions describing the flow is lost. To find the long waves using the conformal mapping $f(\chi) = \alpha\chi + W(\chi)$ with analytic function $W(\chi)$ we must proceed by another way. Suppose $W(\chi)$ is a slow-varying function of $\chi$, i.e. there is a small parameter $\varepsilon$ and the function $W$ depends on $z = x + \mathrm{i}y = \varepsilon\chi$. The value $\varepsilon = 0$ corresponds to the fluid motion with a constant velocity. The Froude number of this flow cannot be arbitrary and has to be equal to one. Only in this case will the shallow-water expansion not be contradictory.

Let us find the solution in the form of the following asymptotic expansion:

$$W(z) = \varepsilon W^{(0)}(z) + \varepsilon^3 W^{(1)}(z) + \cdots. \tag{2.5a}$$

The constants should be found in the form of the analogous expansions:

$$\left.\begin{aligned} \alpha &= 1 + \varepsilon^2\alpha^{(1)} + \varepsilon^4\alpha^{(2)} + \cdots, \\ \delta &= 1 + \varepsilon^2\delta^{(1)} + \varepsilon^4\delta^{(2)} + \cdots, \\ F &= 1 + \varepsilon^2 F^{(1)} + \varepsilon^4 F^{(2)} + \cdots. \end{aligned}\right\} \tag{2.5b}$$

It follows from (2.3), (2.4), that determination of $W(z)$ in the strip of width $\varepsilon$ can be reduced to the following boundary-value problem:

$$\left.\begin{aligned} &\left|\varepsilon\frac{\mathrm{d}W}{\mathrm{d}z} + \alpha\right|^2 = \frac{\delta^2}{\beta - 2\,\mathrm{Im}\, W/F^2} \qquad && (y = \varepsilon), \\ &\mathrm{Im}\, W = 0 && (y = 0). \end{aligned}\right\} \tag{2.6}$$

Substituting the expansions (2.5) into the boundary condition (2.6) and collecting the terms with the same powers of $\varepsilon$, we obtain some differential equations which may be integrated. As a result we obtain the family of cnoidal waves depending on two real parameters $k, \theta$ $(0 \leqslant k \leqslant 1$, $\theta$ is the new small positive parameter):

$$f(\chi) = \chi - \tfrac{4}{3}\theta^2 k^2 \int_0^\chi \mathrm{sn}^2(\theta\chi)\mathrm{d}\chi + O\left(\theta^4\right), \tag{2.7a}$$

and

$$\left.\begin{aligned} \delta &= 1 + 0\,\theta^2 + O\left(\theta^4\right), \\ F &= 1 - \tfrac{2}{3}\left(1 + k^2\right)\theta^2 + O(\theta^4). \end{aligned}\right\} \tag{2.7b}$$

Here sn is the Jacobian elliptic function on modulus $k$. The constant $\alpha$ is undefined, however $\alpha^{(j)}$ may be taken as the arbitrary numbers since the variation of $\alpha$ does not influence the solution (2.7*a*).

It is essential to our further consideration that the function $\mathrm{sn}^2 u$ has two periods:

$$2\mathrm{K}(k), \quad 2\mathrm{i}\mathrm{K}'(k). \tag{2.8}$$

Here and below $\mathrm{K}(k), \mathrm{E}(k)$ are complete elliptic integrals on modulus $k$ of the first and the second kind, respectively and $\mathrm{K}'(k) = \mathrm{K}(k')$, $\mathrm{E}'(k) = \mathrm{E}(k')$, where $k' = (1 - k^2)^{1/2}$.

It follows from (2.7*a*) that the complex velocity $\left(\mathrm{d}f/\mathrm{d}\chi\right)^{-1}$ is periodic along the strip (2.2) on $\varphi$ with a period $2\mathrm{K}(k)/\theta$ and also periodic across the strip (2.2) on $\psi$ with a period $2\mathrm{K}'(k)/\theta$.

The function $f(\chi)$ in (2.7*a*) is not doubly periodic. However, it can be expressed in terms of the function

$$\mathscr{E}(u) \equiv \int_0^u \left(1 - k^2\mathrm{sn}^2 u\right)\,\mathrm{d}u,$$

which does not have (2.8) as the periods (see, for example, Whittaker & Watson 1927), but adding these to the argument the function changes its value by the additive constants

$$2\mathrm{E}(k), \quad \mathrm{i}\frac{2\mathrm{K}'(k)\mathrm{E}(k) - \pi}{\mathrm{K}(k)}.$$

If we choose

$$\alpha = 1 + \tfrac{4}{3}\theta^2\left(\frac{\mathrm{E}(k)}{\mathrm{K}(k)} - 1\right) + O(\theta^4),$$

then the function

$$W(\chi) = f(\chi) - \alpha\chi \tag{2.9}$$

will be periodic with respect to $\varphi$. On the other hand, if

$$\alpha = 1 - \tfrac{4}{3}\theta^2\frac{\mathrm{E}'(k)}{\mathrm{K}'(k)} + O(\theta^4), \tag{2.10}$$

then we obtain the function (2.9) periodic on $\psi$.

Select (2.10), then one term of shallow-water expansion (2.5*a*) gives an approximate solution of Problem 1 which is periodic on $\psi$ with period $2\pi/\lambda_0$, where

$$\lambda_0 \simeq \pi\theta/\mathrm{K}'(k). \tag{2.11}$$

All the rest ot the terms $W_\chi^{(j)}$ $(j > 0)$ are polynomials of $\mathrm{sn}^2(\theta\chi)$. It is possible to find an $\alpha$ such that each term of shallow-water expansion (2.5*a*) will be also periodic on $\psi$ with the same period $2\pi/\lambda_0$.

Below we shall seek $W(\varphi + \mathrm{i}\psi)$ in a class of functions, which are periodic with respect to $\psi$. But the following analysis shows that the exact statement of Problem 1 does not admit solutions of this type.

## 3. Solution periodic across the strip

Let us introduce the new unknown functions

$$P_1(\chi), P_2(\chi), P_3(\chi), \ldots .$$

These functions are connected by the relation

$$P_j(\chi) = W(\chi + \mathrm{i}(2j - 2)) \tag{3.1}$$

with the former unknown $W(\chi)$. Rewrite the boundary condition (2.3) in the form

$$\left(\frac{\mathrm{d}W(\varphi + \mathrm{i})}{\mathrm{d}\varphi} + \alpha\right)\left(\frac{\mathrm{d}\overline{W(\varphi + \mathrm{i})}}{\mathrm{d}\varphi} + \alpha\right) = \frac{\delta^2}{\beta + \mathrm{i}\left(W(\varphi + \mathrm{i}) - \overline{W(\varphi + \mathrm{i})}\right)/F^2}. \tag{3.2}$$

Bottom condition (2.4) at the real axis allows us to apply the symmetry principle and analytically continue function $W(\chi)$ defined in the upper half-plane into the lower half-plane. Hence

$$\overline{W(\varphi+\mathrm{i})}=W(\varphi-\mathrm{i}).$$

Taking this into account, we obtain from (3.1):

$$P_j(\varphi+\mathrm{i}(1-2j))=\overline{W(\varphi+\mathrm{i})},$$
$$P_{j+1}(\varphi+\mathrm{i}(1-2j))=W(\varphi+\mathrm{i}).$$

Thus, (3.2) may be rewritten as

$$\left(\frac{\mathrm{d}P_{j+1}(\varphi+\mathrm{i}(1-2j))}{\mathrm{d}\varphi}+\alpha\right)\left(\frac{\mathrm{d}P_j(\varphi+\mathrm{i}(1-2j))}{\mathrm{d}\varphi}+\alpha\right)$$
$$=\frac{\delta^2}{\beta+\mathrm{i}\left[P_{j+1}(\varphi+\mathrm{i}(1-2j))-P_j(\varphi+\mathrm{i}(1-2j))\right]/F^2}.$$

Taking into account the analyticity of the functions $P_{j+1}, P_j$ we replace the derivatives $\mathrm{d}/\mathrm{d}\varphi$ by $\mathrm{d}/\mathrm{d}\chi$ and obtain the following equation:

$$\left(\frac{\mathrm{d}P_{j+1}}{\mathrm{d}\chi}+\alpha\right)\left(\frac{\mathrm{d}P_j}{\mathrm{d}\chi}+\alpha\right)=\frac{\delta^2}{\beta+\mathrm{i}(P_{j+1}-P_j)/F^2}.$$

Thus we have an infinite system of ordinary differential equations

$$\left.\begin{aligned}\left(\frac{\mathrm{d}P_2}{\mathrm{d}\chi}+\alpha\right)\left(\frac{\mathrm{d}P_1}{\mathrm{d}\chi}+\alpha\right)&=\frac{\delta^2}{\beta+\mathrm{i}(P_2-P_1)/F^2},\\ \left(\frac{\mathrm{d}P_3}{\mathrm{d}\chi}+\alpha\right)\left(\frac{\mathrm{d}P_2}{\mathrm{d}\chi}+\alpha\right)&=\frac{\delta^2}{\beta+\mathrm{i}(P_3-P_2)/F^2},\\ &\cdots\end{aligned}\right\}\tag{3.3}$$

for unknown functions $P_1, P_2, P_3, \ldots$, as the consequence of the problem 1 statement.

Suppose that $W(\chi)$ is a function periodic in a direction across the strip with period $2\pi/\lambda_0$, where $\lambda_0$ is some real number. This function may be represented as Fourier-series in $\psi$. Taking into account the analyticity of $W(\chi)$ we may replace functions sin, cos depending on $\psi$ by functions sinh, cosh depending on $\chi$ in this series. The symmetry condition allows us to eliminate cosh functions. The flat bottom condition provides the reality of the coefficients at sinh. Hence we have

$$W(\chi)=\sum_{j=1}^{\infty}E_j\sinh(j\lambda_0\chi),\quad \operatorname{Im}E_j=0.\tag{3.4}$$

In the case when $\lambda_0=\pi m/n$, where $m, n$ are integers, the system (3.3) is finite, because the set of functions $P_j(\chi)$ will contain periodical subsets consisting of only $n$ different functions:

$$P_1(\chi), P_2(\chi), \ldots, P_n(\chi).$$

Indeed, from the periodicity condition

$$W(\chi)=W(\chi+\mathrm{i}\,2\pi/\lambda_0)=W(\chi+\mathrm{i}\,2n/m),$$

and from (3.1), we have

$$P_{n+1}(\chi)=W(\chi+\mathrm{i}\,2n)=W(\chi)=P_1(\chi).$$

Hence for the rational number $\lambda_0/\pi$ a solution of the Problem 1 periodic across the strip (2.2) is determined by the equality $W = P_1$, where $P_1, P_2, \ldots$ satisfy the system of $n$ ordinary differential equations

$$\left(\frac{\mathrm{d}P_{j+1}}{\mathrm{d}\chi} + \alpha\right)\left(\frac{\mathrm{d}P_j}{\mathrm{d}\chi} + \alpha\right) = \frac{\delta^2}{\beta + \mathrm{i}(P_{j+1} - P_j)/F^2}, \tag{3.5}$$

$$P_{n+1} = P_1 \qquad (1 \leqslant j \leqslant n).$$

## 4. A special case

Before the integration of the system (3.5) we have to fix $\lambda_0$, because this number governs the order of the system. What value of $\lambda_0$ corresponds to the highest waves? For vanishing wave amplitude $\lambda_0 = 0$ and $\lambda_0$ grows with growth of the amplitude ($\theta$ increasing), as evident from (2.11). To what marginal value does the number $\lambda_0$ tend to, when the amplitude reaches its maximum? This value appears to be close to $\pi/3$.

This fact may be shown for solitary waves. Let $u_0, h_0$ be the velocity and the depth of the undisturbed flow at infinity. Then the solitary waves problem is equivalent to Problem 1 with $\alpha = 1$, $\delta = 1$. Its linearization on the trivial solution $W(\chi) \equiv 0$ gives

$$\left.\begin{aligned} &F^2(\operatorname{Im} W)_\psi = \operatorname{Im} W && (\psi = 1), \\ &\operatorname{Im} W = 0 && (\psi = 0). \end{aligned}\right\} \tag{4.1}$$

This approximation is locally valid at infinity, where the free surface is close to the horizontal line $Y = h_0$. We know from Lamb (1932, section 252) that the solution of the problem (4.1) vanishing at the left-hand infinity ($\varphi \to -\infty$) may be obtained in the form

$$\operatorname{Im} W = \mathrm{e}^{\lambda\varphi} \sin \lambda\psi,$$

where $\lambda > 0$ is the arbitrary root of the Stokes equation

$$\frac{\tan\lambda}{\lambda} = F^2. \tag{4.2}$$

Thus we have in a dominant term:

$$W \sim \mathrm{e}^{\lambda_0\chi} \qquad (\varphi \to -\infty), \tag{4.3}$$

where $\lambda_0$ is the smallest root of the equation (4.2). The solution (4.3) is $\psi$-periodic with period $2\pi/\lambda_0$ where $\lambda_0$ is uniquely determined by the Froude number $F$. For the highest solitary wave Evans & Ford (1996) have found the Froude number numerically with high accuracy: $F = 1.290\,890\,53$. Solving (4.2) for this value of $F$ they obtain

$$\lambda_0 = \tfrac{1}{3}\pi \times 1.0052 = 60.314^\circ.$$

Hence for the highest solitary wave, i.e. for the highest gravitational wave of infinite length, $\lambda_0 \simeq \pi/3$. One may expect that for gravitational waves of finite but sufficiently large length this approximate equality will be fulfilled too, but with less accuracy.

Let $\lambda_0 = \pi/3$. We shall seek a $\psi$-periodic solution of Problem 1 with period 6. In this case the order of the system (3.5) is minimal. It consists of three equations only. Now we shall discuss the method of integration of the system (3.5) in this special case.

We have three functions

$$\left.\begin{aligned}
P_1 &= E_1 \sinh \tfrac{1}{3}\pi\chi + E_2 \sinh \tfrac{2}{3}\pi\chi + E_3 \sinh \pi\chi + \dots,\\
P_2 &= E_1 \sinh\left(\tfrac{4}{3}\pi\chi + \mathrm{i}\tfrac{2}{3}\pi\right) + E_2 \sinh\left(\tfrac{2}{3}\pi\chi + \mathrm{i}\tfrac{4}{3}\pi\right) + E_3 \sinh \pi\chi + \dots,\\
P_3 &= E_1 \sinh\left(\tfrac{1}{3}\pi\chi + \mathrm{i}\tfrac{4}{3}\pi\right) + E_2 \sinh\left(\tfrac{2}{3}\pi\chi + \mathrm{i}\tfrac{2}{3}\pi\right) + E_3 \sinh \pi\chi + \dots,
\end{aligned}\right\} \tag{4.4}$$

satisfying the following system of three equations:

$$\begin{aligned}
\left(\frac{\mathrm{d}P_2}{\mathrm{d}\chi} + \alpha\right)\left(\frac{\mathrm{d}P_1}{\mathrm{d}\chi} + \alpha\right) &= \frac{1}{f_1},\\
\left(\frac{\mathrm{d}P_3}{\mathrm{d}\chi} + \alpha\right)\left(\frac{\mathrm{d}P_2}{\mathrm{d}\chi} + \alpha\right) &= \frac{1}{f_2},\\
\left(\frac{\mathrm{d}P_1}{\mathrm{d}\chi} + \alpha\right)\left(\frac{\mathrm{d}P_3}{\mathrm{d}\chi} + \alpha\right) &= \frac{1}{f_3}.
\end{aligned}$$

Here

$$\begin{aligned}
f_1 &= \left[\beta + \mathrm{i}(P_2 - P_1)/F^2\right]/\delta^2,\\
f_2 &= \left[\beta + \mathrm{i}(P_3 - P_2)/F^2\right]/\delta^2,\\
f_3 &= \left[\beta + \mathrm{i}(P_1 - P_3)/F^2\right]/\delta^2.
\end{aligned}$$

The nonlinear system may be reduced to the normal form

$$\left.\begin{aligned}
\frac{\mathrm{d}P_1}{\mathrm{d}\chi} + \alpha &= \frac{f_2}{(f_1 f_2 f_3)^{1/2}},\\
\frac{\mathrm{d}P_2}{\mathrm{d}\chi} + \alpha &= \frac{f_3}{(f_1 f_2 f_3)^{1/2}},\\
\frac{\mathrm{d}P_3}{\mathrm{d}\chi} + \alpha &= \frac{f_1}{(f_1 f_2 f_3)^{1/2}}.
\end{aligned}\right\} \tag{4.5}$$

The main idea for its solution is the introduction of new unknown functions $S_1, S_2, S_3$:

$$\left.\begin{aligned}
S_1 &= \tfrac{1}{2}\left(E_1 \mathrm{e}^{\pi\chi/3} - E_2 \mathrm{e}^{-2\pi\chi/3} + E_4 \mathrm{e}^{4\pi\chi/3} - E_5 \mathrm{e}^{-5\pi\chi/3} + \dots\right),\\
S_2 &= \tfrac{1}{2}\left(-E_1 \mathrm{e}^{-\pi\chi/3} + E_2 \mathrm{e}^{2\pi\chi/3} - E_4 \mathrm{e}^{-4\pi\chi/3} + E_5 \mathrm{e}^{5\pi\chi/3} + \dots\right),\\
S_3 &= E_3 \sinh \pi\chi + E_6 \sinh 2\pi\chi + \dots.
\end{aligned}\right\} \tag{4.6}$$

Comparing these series with (4.4) we conclude that the old variables $P_1, P_2, P_3$ and new variables $S_1, S_2, S_3$ satisfy the relations:

$$\begin{aligned}
P_1 &= S_1 + S_2 + S_3,\\
P_2 &= \mathrm{e}^{\mathrm{i}2\pi/3} S_1 + \mathrm{e}^{\mathrm{i}4\pi/3} S_2 + S_3,\\
P_3 &= \mathrm{e}^{\mathrm{i}4\pi/3} S_1 + \mathrm{e}^{\mathrm{i}2\pi/3} S_2 + S_3.
\end{aligned}$$

We obtain the following system of ordinary differential equations from (4.5) for unknowns $S_1, S_2, S_3$:

$$\left.\begin{aligned}
\frac{\mathrm{d}S_1}{\mathrm{d}\chi} &= \frac{\sqrt{3}}{(F\delta)^2}\frac{S_1}{(f_1 f_2 f_3)^{1/2}},\\
\frac{\mathrm{d}S_2}{\mathrm{d}\chi} &= -\frac{\sqrt{3}}{(F\delta)^2}\frac{S_2}{(f_1 f_2 f_3)^{1/2}},\\
\frac{\mathrm{d}S_3}{\mathrm{d}\chi} &= -\alpha + \frac{\beta}{\delta^2}\frac{1}{(f_1 f_2 f_3)^{1/2}}.
\end{aligned}\right\} \tag{4.7}$$

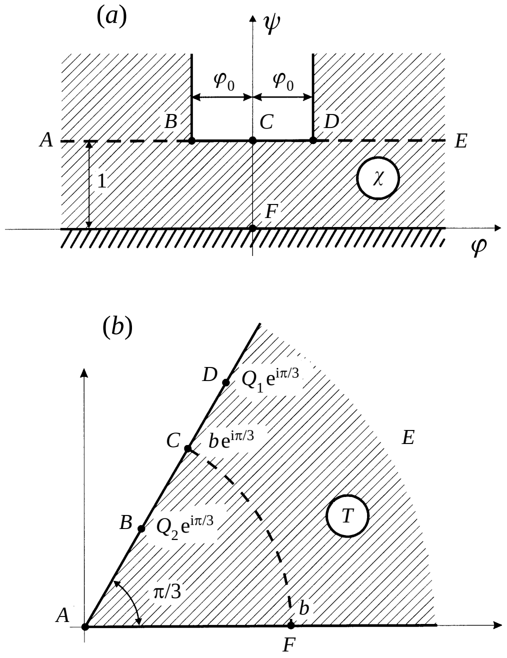


FIGURE 2. Function $T(\chi; b)$ in (4.15*a*) gives the conformal mapping of polygon (*a*), disposed in the complex potential plane, onto wedge (*b*), placed in auxiliary plane $T$.

It follows from (4.6) that

$$S_1(\chi) = -S_2(-\chi), \quad \operatorname{Im} S_1(0) = 0, \quad S_3(0) = 0.$$

Therefore we may consider following data at $\chi = 0$ for the system (4.7):

$$S_1(0) = a, \quad S_2(0) = -a, \quad S_3(0) = 0. \tag{4.8}$$

Here $a$ is some real number.

Combination of the first two equations (4.7) gives the relation

$$\frac{\mathrm{d}\,(S_1\, S_2)}{\mathrm{d}\,\chi} = 0.$$

Hence the system has integral

$$S_1\, S_2 = -a^2.$$

Owing to the presence of this integral, the solution of the initial value problem (4.8) for (4.7) may be readily found.

To define the constants $F, \alpha, \delta$, we substitute the solution into constant pressure boundary condition (2.3) and into normalization conditions (2.1). Only one parameter will be free and all the other ones are determined by it. We shall consider

$$b = \sqrt{3}\frac{a}{F^2\beta}$$

as a free parameter varying from 0 to 1/2.

Let us describe the solutions obtained. Before doing so, we introduce an auxiliary function. Consider the one-parameter quadrangular family in the plane of complex potential $\chi$, shown in figure 2(*a*). Let this family depend on parameter $\varphi_0$ which is a length of segments $BC$ and $DC$. It is not difficult to show that there is a unique quadrangle in this family with

$$\varphi_0(b) = [\mathrm{K}(\mu_*) - \mathrm{E}(\mu_*)]/\mathrm{E}'(\mu_*), \tag{4.9}$$

where

$$\mu_* = \frac{1+b}{1-b}\left(\frac{1-2b}{1+2b}\right)^{1/2},$$

which may be conformally mapped into the wedge

$$0 < |T| < \infty, \quad 0 < \arg T < \pi/3 \tag{4.10}$$

with the correspondence of the points shown on figure 2. The points $A, B, C, D, E, F$, marked on figure 2($a$), are mapped into points $0, Q_2\,\mathrm{e}^{\mathrm{i}\pi/3}, b\,\mathrm{e}^{\mathrm{i}\pi/3}, Q_1\,\mathrm{e}^{\mathrm{i}\pi/3}, \infty, b$ on $T$-plane respectively (see figure 2$b$). Here

$$Q_1 = \tfrac{1}{2} + \frac{(1-4b^2)^{1/2}}{2}, \quad Q_2 = \tfrac{1}{2} - \frac{(1-4b^2)^{1/2}}{2}, \quad 0 < b < \tfrac{1}{2}. \tag{4.11}$$

To find $T$ first unroll wedge (4.10) into the upper half-plane

$$v = T^3, \tag{4.12}$$

and then use the Scwartz–Christoffel integral

$$\chi = \frac{1}{3\kappa}\int_{b^3}^{v} \xi^{-3/2}(\xi + Q_1^3)^{1/2}(\xi + Q_2^3)^{1/2}\mathrm{d}\xi. \tag{4.13}$$

Equating the length of segment $CF$ (see figure 2$a$) to unity we determine:

$$\kappa = \tfrac{2}{3}(1-b)(1+2b)^{1/2}\,\mathrm{E}'(\mu_*). \tag{4.14}$$

Formulae (4.12), (4.13), (4.14) give the dependence $T(\chi; b)$ in an implicit form.

Now we can state the main result. For $\lambda_0 = \pi/3$ there exists a one-parameter family of solutions of system (3.5) determined by formulae:

$$f = \alpha\chi + W = \frac{1}{\pi/3 + \sqrt{3}b}\left(T - \frac{b^2}{T} + \log\frac{T}{b}\right), \tag{4.15$a$}$$

$$\left.\begin{aligned} \alpha &= \frac{\pi/3}{\pi/3 + \sqrt{3}b}, \\ \delta &= \frac{\kappa}{(\pi/3 + \sqrt{3}b)(1-2b)^{1/2}}, \\ F^2 &= \sqrt{3}\frac{1-2b}{\pi/3 + \sqrt{3}b}. \end{aligned}\right\} \tag{4.15$b$}$$

Here $b$ is free parameter ($0 < b < 1/2$).

As indicated on figure 2($b$), function $T$ is real and positive at the bottom, i.e. on ray $AFE$. Therefore, it follows from (4.15$a$) that the function $W$ also is real on $AFE$ and flat bottom condition (2.4) is fulfilled.

Let us next show that on $BCD$ (see figure 2$a$) the constant pressure condition is also fulfilled. As illustrated on figure 2($b$), we have

$$T = Q\,\mathrm{e}^{\mathrm{i}\pi/3} \tag{4.16}$$

on $BCD$. Here $Q$ is a real parameter varying from $Q_2$ to $Q_1$. Therefore from (4.15$a$)

it follows that

$$f = \alpha\chi + W = \frac{1}{\pi/3 + \sqrt{3}b}\left(Q\mathrm{e}^{\mathrm{i}\pi/3} - \frac{b^2}{Q}\mathrm{e}^{-\mathrm{i}\pi/3} + \mathrm{i}\frac{\pi}{3} + \log\frac{Q}{b}\right) \tag{4.17}$$

on $BCD$. Substituting the imaginary part of this equality into the boundary condition (2.3) and taking into account the formulae (4.15$b$) we obtain

$$\left|\frac{\mathrm{d}f}{\mathrm{d}Q}\right|^2 \left|\frac{\mathrm{d}Q}{\mathrm{d}\chi}\right|^2 = \frac{\kappa^2}{\left(\pi/3 + \sqrt{3}b\right)^2}\frac{1}{1 - Q - b^2/Q}. \tag{4.18}$$

It is necessary to prove that this is an identity. The first factor on the left-hand side of this equality is determined from (4.17):

$$\left|\frac{\mathrm{d}f}{\mathrm{d}Q}\right|^2 = \frac{1}{(\pi/3 + \sqrt{3}b)^2}\left(1 + \frac{1}{Q^2} - \frac{b^2}{Q^2} + \frac{1}{Q} + \frac{b^4}{Q^4} + \frac{b^2}{Q^3}\right).$$

To find the second factor we differentiate the integral (4.13) with respect to $v$:

$$\frac{\mathrm{d}v}{\mathrm{d}\chi} = 3\kappa\frac{v^{3/2}}{\left(v^2 + (1 - 3b^2)v + b^6\right)^{1/2}}.$$

Using the equality $v = -Q^3$, derived from (4.12), (4.16), we obtain

$$\left|\frac{\mathrm{d}Q}{\mathrm{d}\chi}\right|^2 = \kappa^2\frac{Q^5}{-Q^6 + (1 - 3b^2)Q^3 - b^6}.$$

Therefore, the equality (4.18) is rewritten in the form

$$\left(1 + \frac{1}{Q^2} - \frac{b^2}{Q^2} + \frac{1}{Q} + \frac{b^4}{Q^4} + \frac{b^2}{Q^3}\right)\left(\frac{1}{Q} - 1 - \frac{b^2}{Q^2}\right) = -1 + \frac{1 - 3b^2}{Q^3} - \frac{b^6}{Q^6}.$$

Obviously it is an identity.

Let us prove that the function

$$W(\varphi + \mathrm{i}\psi) = \frac{1}{(\pi/3 + \sqrt{3}b)}\left(T - \frac{b^2}{T} + \log\frac{T}{b} - \frac{\pi}{3}\chi\right)$$

is periodic with respect to $\psi$ for $|\varphi| < \varphi_0$ with the period 6. Consider a function giving the conformal mapping of one polygon into the other. The analytical continuation of this function according to the symmetry principle gives the function realizing conformal mapping into two new polygons. New polygons are obtained by the mirror reflection of the initial polygons relative to those sides across which the analytic continuation is realized. The function $T$ conformally maps the polygon shown on figure 2($a$) into the polygon, shown on figure 2($b$). Analytically continuing $T$ through $BCD$ we see that when $\psi$ increases to 6, the point $T$ completes one turn around $A$ and returns to the initial position. So $\arg T$ increment is equal to $2\pi$. Hence the function $\log T - (\pi/3)\chi$ and consequently $W$ will be periodic on $\psi$ with period 6.

In the vicinity of point $D$ the conformal mapping admits the representation

$$f(\chi) - f(\varphi_0 + \mathrm{i}) \simeq \text{const}\left(1 + \frac{b^2}{Q_1}\mathrm{e}^{-2\mathrm{i}\pi/3} + \frac{1}{Q_1}\mathrm{e}^{-\mathrm{i}\pi/3}\right)(\chi - \varphi_0 - \mathrm{i})^{2/3}\ (\text{const} > 0).$$

Hence $D$ is the singular point. Here the upper fluid boundary has a 120° corner. An analogous conclusion is valid for the point $B$. The section $BCD$ on the upper strip

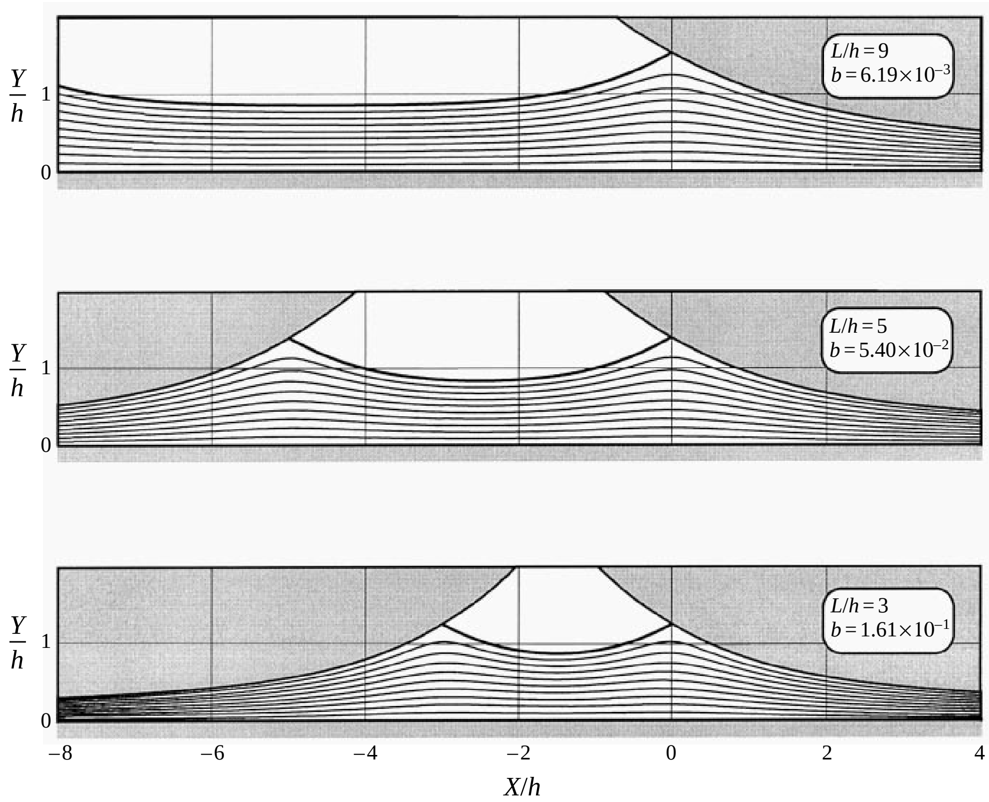


FIGURE 3. The streamlines for a one-parameter family of steady flows of an inviscid incompressible heavy fluid are depicted. The upper fluid boundary consists of the two curvilinear roofs and of the free surface. From below the fluid is bounded by the flat bottom. The horizontal and vertical scales are equal. Letters $h$ and $L$ designate respectively the mean depth under free surface and distance between two corner points on the upper fluid boundary. Free parameter $b$ varies from $b = 0$ ($L/h = \infty$) up to $b = 1/2$ ($L/h = 0$).

boundary corresponds to a free boundary, but on the remaining parts, namely on the rays $DE$ and $BA$ (shown by a broken line on figure 2$a$) the constant pressure condition is not fulfilled. The streamlines $DE$ and $BA$ may be considered as curvilinear walls. The asymptotic analysis of the solution as $|\varphi| \to 0$ gives the following behaviour of these walls:

$$X \sim |\varphi|^{2/3}, \quad Y \sim |\varphi|^{-1/3}.$$

The family of the exact solutions of the Euler equations that has been constructed describes the heavy fluid flows with a partially free surface, which are distinguished from the water waves. The fluid moves under two curvilinear roofs, which are symmetrically located and approach the bottom as $|\varphi| \to 0$. The streamlines and the roof shape for flows of this family are shown on figure 3 with the same horizontal and vertical scales. The mean depth under the free surface $BCD$ is taken as a unit of measurement. The fluid emerges from beneath the left-hand roof and passes into the right-hand roof. With variation of $b$ the form of the roofs and the distance between them vary. The limit case $b \to 0$ discussed previously by Karabut (1996) corresponds to an infinite distance. In another case $b \to 1/2$, the roofs approach each other and the free surface disappears in this limit.

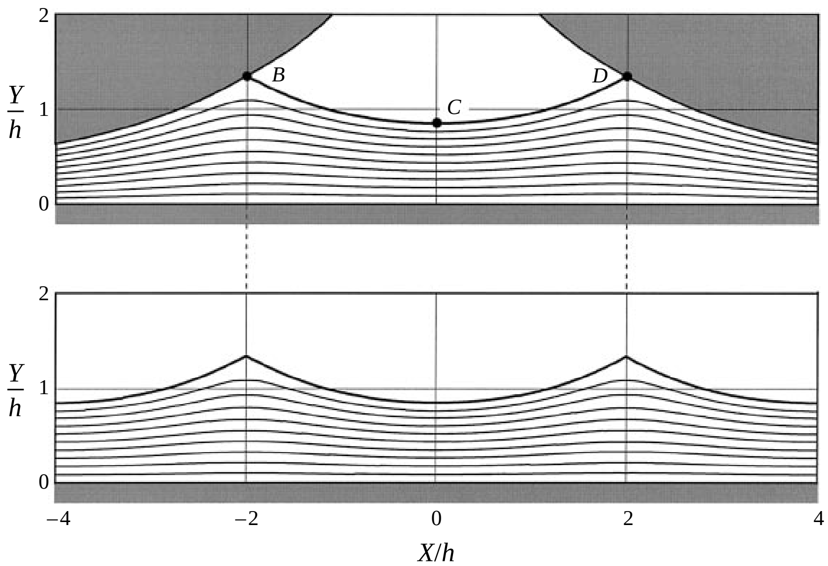


FIGURE 4. The streamlines for one of the flows constructed are displayed at the upper part. The flow in the region under free surface $BCD$ is close to the highest gravity wave. The picture in (*b*) is constructed by periodically pasting removed copies of this region next to each other.

## 5. Characteristics of the highest waves

In this section, comparing our results with the precise numerical results by Williams (1981), we show that the flows constructed are close to the highest gravity waves.

Let $\Gamma$ be the part of the free surface for the highest wave between two neighbouring crests. Let us drop perpendiculars from these crests to the bottom. The region bounded by these two perpendiculars, the free surface $\Gamma$ and the bottom will be denoted by $\Omega$. Also let $\Gamma^{(0)}$ be the free surface for the above-constructed flows. Let us drop perpendiculars from the points of intersection of the curve roofs with $\Gamma^{(0)}$ (these are points $B$ and $D$ on figure 4) to the bottom. The wave problem is equivalent to the following: to find the region $\Omega$ and the solution of the boundary-value problem in $\Omega$ satisfying the constant pressure boundary condition at the free surface, the even bottom condition at the bottom and the velocity vector horizontality condition on the perpendiculars. We have found the region $\Omega^{(0)}$ for which the first two conditions are fulfilled exactly and the third condition is fulfilled approximately. The latter is evident from figure 5, where plots of the horizontal $u_X$ and vertical $u_Y$ velocities on a vertical line beneath the highest point of the free surface are represented. We notice that $u_Y$ is far less than $u_X$. Consequently, the above mentioned boundary-value problem is solved approximately. Therefore it should be expected that approximate equalities

$$\Gamma^{(0)} \simeq \Gamma, \quad \Omega^{(0)} \simeq \Omega$$

will be valid and the gravity wave parameters in $\Omega$ will differ slightly from the corresponding quantities in $\Omega^{(0)}$. These expectations are realized. The gravitational wave solution can be approximately constructed by the manner indicated in figure 4, i.e. by periodical disposition of copies of the region $\Omega^{(0)}$ along the $X$-axis.

The problem of calculation of properties for the limiting wave is simplified because the free surface shape for the constructed solutions is given by simple formulae. Taking the real and imaginary parts of (4.17) we obtain the following parametric

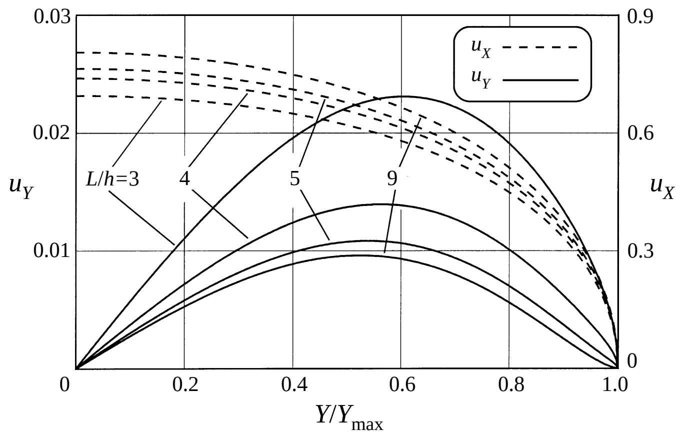


FIGURE 5. Horizontal velocity $u_X = \Phi_X/(gh)^{1/2}$ (right-hand axis) and vertical velocity $u_Y = \Phi_Y/(gh)^{1/2}$ (left-hand axis) on the perpendicular dropped from the highest point $B$ ($Y = Y_{max}$) to the bottom ($Y = 0$). This plots is a measure of the accuracy of the approximation.

representation:

$$X = \gamma\left(\tfrac{1}{2}Q - \frac{b^2}{2Q} + \log\frac{Q}{b}\right), \quad Y = \gamma\left[\frac{\sqrt{3}}{2}\left(Q + \frac{b^2}{Q}\right) + \tfrac{1}{3}\pi\right], \quad (Q_2 \leqslant Q \leqslant Q_1), \tag{5.1}$$

where $Q_1$, $Q_2$ are determined by (4.11) and

$$\gamma = h_0/\big(\pi/3 + \sqrt{3}b\big).$$

Values $Q_2, b, Q_1$ of $Q$ correspond to points $B, C, D$ respectively. Using these formulae (5.1) we can represent all wave properties as functions of $b$.

Let us compare, for example, the dependence of the wave amplitude on the wavelength given by our theory with the dependence obtained by Williams. The wavelength $L$, as the distance between two neighbouring crests, may be approximately estimated, if we assign to the points $B$ and $D$ in figure 4 the two neighbouring crests of the exact free wave surface. Hence, we have

$$L \simeq X_D - X_B.$$

Substituting $Q_2$ and $Q_1$ in (5.1), we obtain

$$L \simeq \gamma\hat{L}, \quad \hat{L} = \big(1 - 4b^2\big)^{1/2} + \log\frac{1 + \big(1 - 4b^2\big)^{1/2}}{1 - \big(1 - 4b^2\big)^{1/2}}. \tag{5.2}$$

In a similar manner, the amplitude of a gravitational wave

$$H = Y_{crest} - Y_{trough},$$

may be estimated:

$$H \simeq Y_D - Y_C.$$

Substitution of $Q_1$ and $b$ in (5.1) gives

$$H \simeq \gamma\hat{H}, \quad \hat{H} = \frac{\sqrt{3}}{2}(1 - 2b). \tag{5.3}$$

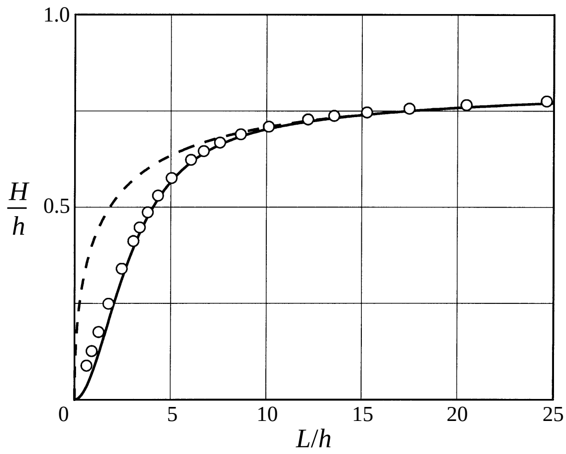


FIGURE 6. The highest wave amplitude $H/h$ as a function of wavelength $L/h$. The circles correspond to Williams's data. The continuous curve illustrates the dependences (5.2), (5.3), (5.4) given by our theory. The dashed curve corresponds to the simplified formula (5.12) suitable for long waves.

The mean depth of the gravity wave is

$$h = S/L,$$

where

$$S = \int_{\Gamma} Y \, \mathrm{d}X$$

is the area of $\Omega$, which may be estimated by the approximate equality

$$S \simeq \int_{\Gamma^{(0)}} Y \, \mathrm{d}X.$$

Using (5.1) we obtain the following approximate formulae for the mean depth:

$$\left.\begin{aligned} & h \simeq \gamma \hat{h}, \quad \hat{h} = \hat{S}/\hat{L}, \\ & \hat{S} = \left(\tfrac{5}{4}\sqrt{3} + \tfrac{1}{3}\pi\right)\left(1 - 4b^2\right)^{1/2} + \left(\tfrac{1}{2}\sqrt{3}b^2 + \tfrac{1}{3}\pi\right)\log\left(\frac{1 + \left(1 - 4b^2\right)^{1/2}}{1 - \left(1 - 4b^2\right)^{1/2}}\right). \end{aligned}\right\} \qquad (5.4)$$

Formulae (5.2), (5.3), (5.4) give the dimensionless amplitude $H/h$ as a function of the dimensionless wavelength $L/h$ in parametric form. This dependence is shown as a continuous line at figure 6. The circles correspond to 21 points computed by Williams.

Similarly we may obtain the formulae for other parameters. The dimensionless wave velocity is given by the formula

$$c = \frac{1}{L(gh)^{1/2}} \int_{\Gamma} \Phi_X \mathrm{d}X = \frac{\Delta\Phi}{L(gh)^{1/2}}. \qquad (5.5)$$

Here $\Delta\Phi$ is the difference of velocity potential values in two neighbouring crests. This quantity may be approximated as

$$\Delta\Phi \simeq \Phi_D - \Phi_B,$$

where $\Phi_D, \Phi_B$ are the values of the velocity potential at $D$ and $B$. Taking into account the relations

$$\Phi_D = \Psi_0\varphi_0, \quad \Phi_B = -\Psi_0\varphi_0,$$

and using (4.9) we find $\Delta\Phi$. Substituting the above formulae for $L, h$ into (5.5) gives an approximation for the dimensionless wave velocity:

$$c \simeq 2 \times 3^{1/4} \frac{\kappa\varphi_0}{\hat{L}\,\hat{h}^{1/2}}. \tag{5.6}$$

Formulae (5.2), (5.4), (5.6), (4.9) allow us to determine the wave velocity as a function of the wavelength.

The wave momentum per unit length along the $X$-axis is given by the formula

$$I = \frac{1}{Lh(gh)^{1/2}} \iint_{\Omega} \left(c(gh)^{1/2} - \Phi_X\right) \mathrm{d}X\,\mathrm{d}Y. \tag{5.7}$$

Transformation of the integral over the region to the integral along the contour gives

$$I = c - \frac{1}{Lh(gh)^{1/2}} \int_{\Gamma} Y\,\mathrm{d}\Phi.$$

An approximation for the momentum may be obtained using the above approximate formulae for $L, h, c$ and change to the integral along $\Gamma$ from the integral along $\Gamma^{(0)}$:

$$\int_{\Gamma} Y\,\mathrm{d}\Phi \simeq \Psi_0 \int_{Q_2}^{Q_1} Y \frac{\mathrm{d}\varphi}{\mathrm{d}Q}\mathrm{d}Q.$$

Introducing the variable $\tau = Q + b^2/Q$ allows us to calculate

$$I \simeq \frac{2 \times 3^{1/4}\kappa\varphi_0\left(\hat{h} - \pi/3\right) - 3^{3/4}J(b)}{\hat{L}\,\hat{h}^{3/2}}, \tag{5.8}$$

where

$$J(b) = \int_{2b}^{1} \tau \left(\frac{(1-\tau)(\tau^2+\tau+1-3b^2)}{\tau^2-4b^2}\right)^{1/2} \mathrm{d}\tau. \tag{5.9}$$

The dimensionless potential energy for the gravity wave is

$$V = \frac{1}{Lh^2} \iint_{\Omega} \left(Y - \frac{h}{2}\right) \mathrm{d}X\,\mathrm{d}Y$$

or

$$V = \frac{1}{2Lh^2} \int_{\Gamma} Y^2 \mathrm{d}X - \tfrac{1}{2}.$$

Again suppose that the last integral may be approximately taken over $\Gamma^{(0)}$. Using (5.1) and taking into account the above formulae for $L, h$, we obtain

$$V \simeq \frac{1}{\left(\hat{L}\hat{h}\right)^2} \left[ \left(\tfrac{3}{4}b^2 - \tfrac{3}{8}b^4\right) \log^2 \frac{1+\left(1-4b^2\right)^{1/2}}{1-\left(1-4b^2\right)^{1/2}} \right.$$
$$\left. + \left(-\tfrac{1}{8}b^2 + \tfrac{1}{2}\right)\left(1-4b^2\right)^{1/2} \log \frac{1+\left(1-4b^2\right)^{1/2}}{1-\left(1-4b^2\right)^{1/2}} + (1-4b^2)\left(b^2 - \tfrac{59}{32}\right) \right]. \tag{5.10}$$

For calculation of the dimensionless kinetic energy per unit length along the $X$-axis

$$E = \frac{1}{2Lgh^2} \iint_{\Omega} \left[ \left(\Phi_X - c(gh)^{1/2}\right)^2 + \Phi_Y^2 \right] \mathrm{d}X\,\mathrm{d}Y$$

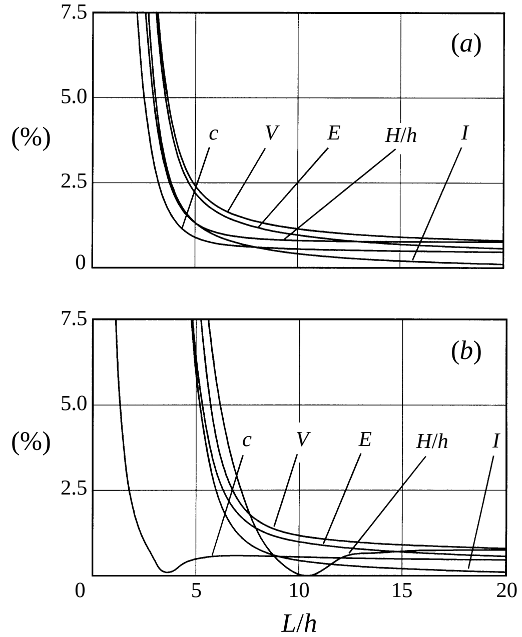


FIGURE 7. The relative error of the computation of amplitude $H/h$, wave velocity $c$, momentum $I$, potential $V$ and kinetic $E$ energies as a function of wavelength $L/h$ are plotted: (*a*) the formulae (5.3), (5.4), (5.5), (5.7), (5.10); (*b*) the simplified formulae (5.12), (5.13), (5.14), (5.15) for long waves.

it will suffice to use identity $E = cI/2$ (Longuet-Higgins 1984) and approximate formulae for velocity $c$ (5.6) and momentum $I$ (5.8). As a result we obtain the approximated formula for the kinetic energy.

The estimation of the accuracy of all obtained formulae carried out from comparison of our results with Williams's (1981) results is represented in figure 7(*a*). We notice that the solutions constructed describe the gravity waves well for long to medium waves ($L/h > 4$).

Parameter $b$ varies from 0 to 0.1 with variation of $L/h$ from $\infty$ to 4. Therefore this parameter is small over our theory efficiency range. This enables us to construct simplified formulae involving only elementary functions. To do this, we neglect $b^j$, $b^j \log b$ $(j > 0)$ in comparison with $b^0$, $\log b$. The new formulae are suitable for long waves, although for medium waves they are less accurate.

For example, from (5.2), (5.3), (5.4) we have the simplified formulae:

$$\hat{L} \simeq 1 - 2\log b, \quad \hat{H} \simeq \tfrac{1}{2}\sqrt{3}, \quad \hat{S} \simeq \tfrac{5}{4}\sqrt{3} + \tfrac{1}{3}\pi - \tfrac{2}{3}\pi \log b. \tag{5.11}$$

Therefore the parametric dependence of the dimensionless amplitude on the dimensionless wavelength is

$$\frac{H}{h} \simeq \tfrac{1}{2}\sqrt{3}\,\frac{1 - 2\log b}{\tfrac{5}{4}\sqrt{3} + \tfrac{1}{3}\pi - \tfrac{2}{3}\pi\log b}, \quad \frac{L}{h} \simeq \frac{(1 - 2\log b)^2}{\tfrac{5}{4}\sqrt{3} + \tfrac{1}{3}\pi - \tfrac{2}{3}\pi\log b}. \tag{5.12}$$

This dependence is shown by the dashed line in figure 6.

Analogous simplified formula may be obtained for the wave velocity. Taking into account the asymptotic behaviour of elliptic integrals:

$$\mathrm{E}(\mu_*) \simeq 1, \quad \mathrm{K}(\mu_*) \simeq \log \frac{4}{(1 - \mu_*^2)^{1/2}} \quad \text{as} \quad \mu_* \to 1$$

we obtain from (4.9) and (4.14)

$$\kappa\varphi_0 \simeq \tfrac{2}{3}(\log 2 - 1) - \log b.$$

Substituting this and (5.12) into (5.6) gives

$$c \simeq 2 \times 3^{1/4} \frac{\tfrac{2}{3}(\log 2 - 1) - \log b}{(1 - 2\log b)^{1/2}\left(\tfrac{5}{4}\sqrt{3} + \tfrac{1}{3}\pi - \tfrac{2}{3}\pi\log b\right)^{1/2}}. \tag{5.13}$$

Calculating the limiting value of the integral (5.9)

$$\lim_{b\to 0} J(b) = J_* = \int_0^1 \left(1 - \tau^3\right)^{1/2} \mathrm{d}\tau \simeq 0.841\,31$$

we obtain the momentum formula from (5.8):

$$I \simeq 3^{3/4} \frac{\left[\tfrac{5}{3}(\log 2 - 1) - J_*\right] - \left(\tfrac{5}{2} - 2J_*\right)\log b}{\left(\tfrac{5}{4}\sqrt{3} + \tfrac{1}{3}\pi - \tfrac{2}{3}\pi\log b\right)^{3/2}(1 - 2\log b)^{1/2}}. \tag{5.14}$$

Similarly for potential energy we obtain

$$V \simeq \frac{-59/32 - \log b}{\left(\tfrac{5}{4}\sqrt{3} + \tfrac{1}{3}\pi - \tfrac{2}{3}\pi\log b\right)^2}. \tag{5.15}$$

instead of formula (5.10).

The comparison of data obtained from the simplified formulae with Williams's data is given on figure 7(*b*).

The accuracy of our approximation increases with a wavelength increase. Let us evaluate the accuracy for the longest waves. With $L/h \to \infty$ we have the following asymptotes:

$$h \sim h_*, \qquad H \sim H_*, \qquad c \sim c_*,$$

$$I \sim I_*/(L/h), \quad V \sim V_*/(L/h), \quad E \sim E_*/(L/h).$$

Here the constants $H_*/h_*, c_*, I_*, V_*, E_*$ may be calculated approximately using formulae (5.12), (5.13), (5.14), (5.15) of our theory. Corresponding values are listed in table 1. For a solitary wave of maximal amplitude $H_\infty$ in water of depth $h$ Williams has computed: the velocity $c_\infty$, the momentum $I_\infty$, the potential energy $V_\infty$ and the kinetic energy $E_\infty$. The following equalities have to be fulfilled:

$$H_*/h_* = H_\infty/h, \quad c_* = c_\infty, \quad V_* = V_\infty, \quad E_* = E_\infty, \tag{5.16}$$

but

$$I_* \neq I_\infty.$$

The appropriate equality for the momentum may be found if we define the momentum differently than in (5.7):

$$\tilde{I} = \frac{1}{Lh(gh)^{1/2}} \iint_\Omega (u_0 - \Phi_X)\mathrm{d}X\mathrm{d}Y = I + \frac{h}{L}C, \quad C = \frac{1}{h(gh)^{1/2}} \int_{-L/2}^{L/2} (u_0 - \Phi_X)\mathrm{d}X.$$

Here $u_0$ is the trough velocity. Taking into account that

$$\lim_{L\to\infty} (L/h)\tilde{I} = I_\infty,$$

| The data of this paper | The Williams data | The relative error |
|---|---|---|
| $H_*/h_* = 3^{3/2}/(2\pi) = 0.8270$ | $H_\infty/h = 0.8332$ | 0.75% |
| $c_* = 3^{3/4}\pi^{-1/2} = 1.2861$ | $c_\infty = 1.2909$ | 0.37% |
| $V_* = 27/(2\pi^3) = 0.4354$ | $V_\infty = 0.4377$ | 0.51% |
| $E_* = 81(5/4 - J_*)/(2\pi^3) = 0.5338$ | $E_\infty = 0.5350$ | 0.22% |
| $I_* = 3^{13/4}\pi^{-5/2}(5/4 - J_*) = 0.8302$ | $I_\infty - C_\infty = 0.8289$ | 0.15% |

TABLE 1. The comparison of our results with Williams's data for a solitary wave.

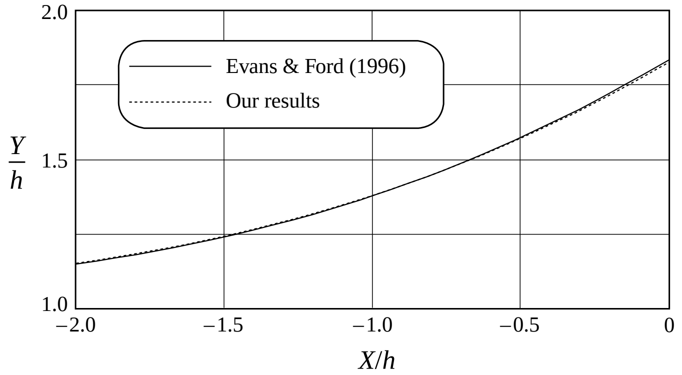


FIGURE 8. The left half of the free boundary of the highest solitary wave. The continuous line coresponds to precise results of Evans & Ford (1996). The dashed line is obtained from formulae: $X/h = (3/2\pi)(Q - 1 + 2\log Q)$, $Y/h = (3^{3/2}/2\pi)Q + 1$ $(0 \leqslant Q \leqslant 1)$.

we see that the following equality has to be fulfilled:

$$I_* = I_\infty - C_\infty. \tag{5.17}$$

The value

$$C_\infty = \frac{1}{h(gh)^{1/2}} \int_{-\infty}^{\infty} (u_\infty - \Phi_X)\mathrm{d}X$$

for the solitary wave has been computed by Williams. The equalities (5.16), (5.17) are approximately fulfilled as it follows from table 1. The error of our theory for the case of very long waves is less than 1%. The mean error is about 0.5%. The free boundary for long waves can be found with the same good accuracy. This is evident from figure 8, where a comparison with the result of Evans & Ford (1996) for the highest solitary waves is given.†

We can now make an analytical investigation of the highest amplitude water waves. We exploit this possibility to study other problems. For example, if a boat is rolling and pitching on waves in shallow water, a situation may happen where the boat strikes the bottom. This possibility also occurs for boats at anchor. What is a safe depth? To the author's knowledge this interesting question has not been discussed in the literature. Assuming that a boat does not perturb the water wave boundary, we may solve this problem within the framework of our theory. Let various finite-amplitude waves be generated in a channel of depth $h$. The smallest value of $h_0/h$ ($h_0$ is the

† In the paper of Evans & Ford (1996) in table 3 a typographic error occurred. It is necessary to read $b_{14} = -8.1152\cdots$ instead of $b_{14} = -81.152\cdots$.

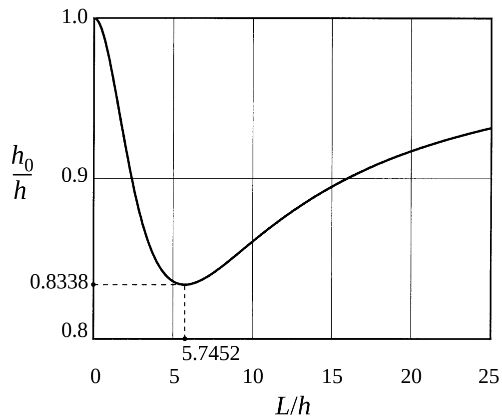


FIGURE 9. The ratio of the trough depth $h_0$ to the mean depth $h$ for gravity highest waves as a function of the wavelength.

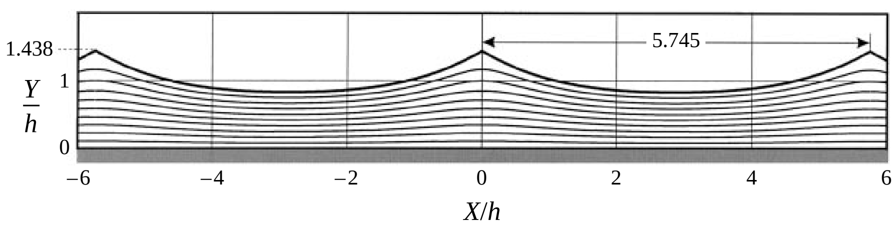


FIGURE 10. For this wave the distance between the lowest point of the free surface and the bottom is minimal among all gravity waves.

trough depth) characterizes the most dangerous wave. Mathematically, the problem of recognizing the most dangerous wave is equivalent to finding the minimum of $h_0/h$. With fixed wavelength, the value $h_0/h$ decreases as the amplitude increases. Therefore, the minimum $h_0/h$ appears among the highest waves. The dependence of $h_0/h$ on $L/h$ derived from (5.2), (5.3), (5.4) is shown in figure 9. For the longest and the shortest waves $h_0/h = 1$, but for the remaining waves $h_0/h$ is less than unity. As illustrated in figure 9, the curve has the well-defined minimum $(h_0/h)_{\min} = 0.8338$. It is attained with $L/h = 5.745$, i.e. in the region where our theory is effective. The most dangerous wave is shown in figure 10.

Notice that our theory gives the quantity $h_0/h$ more accurately than the other quantities. For example, the formulae (5.4) with $b = 0$ (this value corresponds to a solitary wave) gives the exact value $h_0/h = 1$. Comparing $(h_0/h)_{\min}$ with a maximal amplitude of the solitary wave $H_\infty/h$ given in table 1 we come to a hypothesis:

$$(h_0/h)_{\min} = H_\infty/h. \tag{5.18}$$

The most precise value of $H_\infty/h = 0.833\,199\,179$ was found by Evans & Ford (1996). It should be interesting to find $(h_0/h)_{\min}$ with higher accuracy and verify if this coincidence has more substance. If the equation (5.18) is fulfilled then the distance between the highest point of the free surface attained under the propagation of various gravity waves and the corresponding lowest point is exactly equal to the depth.

We have explicit formulae for several integral quantities. Also it is possible to give the explicit formulae for local values such as velocity and pressure. For practical applications it would useful to have expressions for velocity and pressure along the bottom. Denote $A = 2P/\rho u_0^2$, where $P, \rho, u_0$ are the pressure on the bottom, fluid density, velocity at lowest point of free surface respectively. It follows from the Bernoulli integral that

$$A = 1 + \frac{2}{F^2} - \delta^2 \left| \frac{\mathrm{d}\chi}{\mathrm{d}f} \right|^2 .$$

Using the above formulae (4.15) we obtain

$$A(T,b) = 1 + \frac{2}{F^2} - \frac{1}{1-2b} \frac{T^4 - T^3 + T^2(1-b^2) - Tb^2 + b^4}{T(T^2 + T + b^2)}. \tag{5.19}$$

On the bottom $AFE$ the variable $T$ is positive and for point $F$ under the trough of the free surface we have $T = b$ (see figure 2). The pressure increases with increasing $T - b$ and peaks at some bottom point $T = T^*$. This point is distinct from the point located under a 'crest' of the constructed flows. However, the difference of pressure at these points is less than our theory error. Therefore, the point $T^*$ will be considered to correspond to the point under the water wave crest. This assumption simplifies the formulae without increasing its error. Finding the maximum of (5.19) we get

$$T^* = \tfrac{1}{2} \left( [3(1-b^2)]^{1/2} - 1 + \left(4 - 2[3(1-b^2)]^{1/2} - 7b^2\right)^{1/2} \right).$$

The bottom pressure $P$ as a function of the coordinate $X$ $(0 \leqslant X \leqslant L/2)$ is given in parametric form by the following formulae:

$$\left.\begin{aligned} \frac{P}{\rho g h} &= \sqrt{3}\frac{1-2b}{2\hat{h}} A(T,b), \\ \frac{X}{L} &= \frac{1}{\hat{L}}\left(T - \frac{b^2}{T} + \log\frac{T}{b}\right), \qquad T \in [b, T^*]. \end{aligned}\right\} \tag{5.20}$$

The contours of bottom pressure obtained from this formulae are represented in figure 11.

If we substitute $T = b$ and $T = T^*$ into (5.20) then we obtain approximate expressions for minimal and maximum bottom pressure for the highest water wave. The wavelength-dependence of this is shown by full lines in figure 12(*a*). The dotted lines correspond to the presumed behaviour outside the application area of our theory. The least dimensionless pressure 0.8583 is reached with $L/h = 7.8545$. The formulae are too cumbersome here, but they can be simplified for long waves, when parameter $b$ is small. Taking into consideration that

$$\lim_{b\to 0} A(b,b) = 2\sqrt{3}\pi, \qquad \lim_{b\to 0} A(T^*,b) = 4 + 2\sqrt{3}\left(\tfrac{1}{9}\pi - 1\right)$$

we obtain

$$\frac{P_{min}}{\rho g h} \simeq \frac{\pi}{3} \frac{1 - 2\log b}{\frac{5}{4}\sqrt{3} + \frac{1}{3}\pi - \frac{2}{3}\pi \log b}, \qquad \frac{P_{max}}{\rho g h} \simeq k \frac{P_{min}}{\rho g h}, \tag{5.21}$$

where

$$k = 1 + \frac{6\sqrt{3} - 9}{\pi} \simeq 1.4431.$$

The dependences corresponding to these simplifying formulae are shown by broken lines at figure 12(*a*). It follows from (5.21), that the ratio of the maximum bottom pressure to the minimum one has the approximately constant value: $P_{max}/P_{min} \simeq k$. For the highest solitary wave we have $P_{min} = \rho g h$, hence we obtain the following

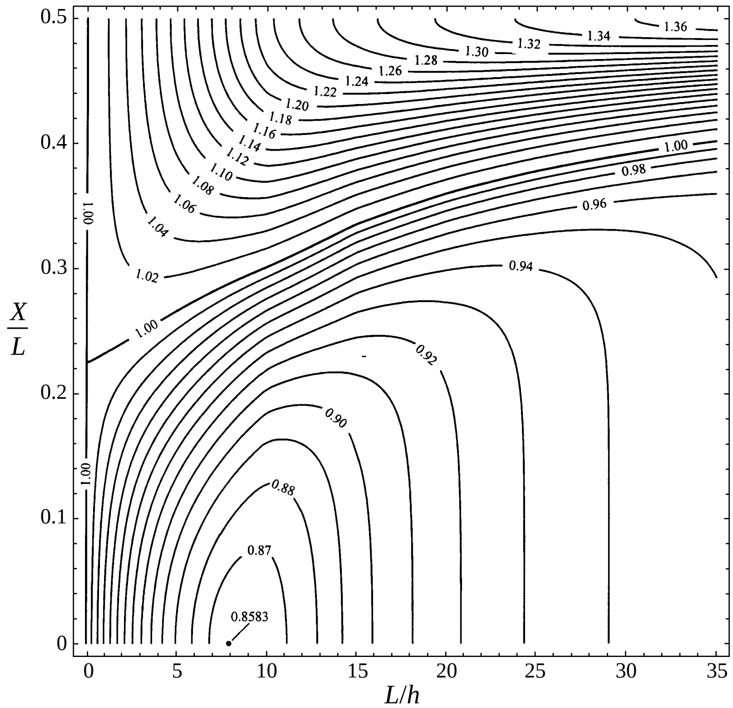


FIGURE 11. The contours of dimensionless bottom pressure $P/\rho g h$ in the plane $X, L$. $X$ is the horizontal coordinate. $L$ is the wavelength. The values $X/L = 0$ and $X/L = 0.5$ correspond to the bottom points under the trough and crest respectively.

maximum bottom pressure: $P_{\max}/\rho g h \simeq 1.4431$. This differs by 0.78% from the exact value 1.4544 numerically calculated by Evans & Ford (1996).

Designate by $u$ the dimensionless bed velocity $\Phi_X/(gh)^{1/2}$. From (4.15) we have

$$u = \left( \frac{\sqrt{3}}{\hat{h}} \frac{T^4 - T^3 + T^2(1-b^2) - Tb^2 + b^4}{T(T^2 + T + b^2)} \right)^{1/2}.$$

Substitutioning here $T = b$ and $T = T^*$ we obtain the velocity under the trough and the crest correspondingly. Simplified formulae for these quantities for long waves are

$$u_{\text{trough}} \simeq \frac{3^{1/4}}{\hat{h}^{1/2}}, \quad u_{\text{crest}} \simeq \frac{\left(6 - 3\sqrt{3}\right)^{1/2}}{\hat{h}^{1/2}}.$$

In the limit of a solitary wave we obtain from here $u_{\text{trough}} \simeq 3^{3/4}\pi^{-1/2} = 0.8270$ and $u_{\text{crest}} \simeq \left((18 - 9\sqrt{3})/\pi\right)^{1/2} = 0.8761$. These differ by 0.75% and 0.66% respectively from the exact values 0.8332 and 0.8704.

In practice, the velocity $U = c - u$ taken in the coordinate system where the mean fluid velocity equals zero is of more interest. Figure 12($b$) displays the wavelength dependence of maximum and minimum of this quantity:

$$U_{\max} = c - u_{\text{crest}}, \quad U_{\min} = c - u_{\text{trough}}.$$

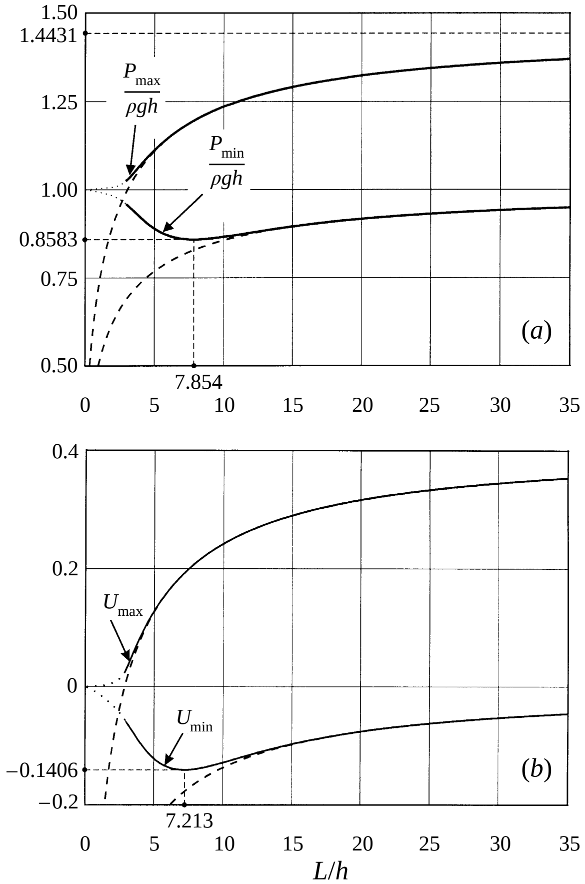


FIGURE 12. (*a*) The maximum and minimum bottom pressure as a function of the wavelength. (*b*) The maximum and minimum bottom velocity. The broken curves correspond to simplified formulae for long waves.

## 6. Discussion

The formulae derived above provide approximations to the highest waves which may be of practical value where analytical expressions, or rapid computations, are required for the highest waves with lengths greater than about 4 times the depth.

Our hypothesis on the $\psi$-periodicity of $W(\chi)$ leads to a contradiction. Hence, the exact solution of Problem 1 does not satisfy the periodicity condition across the strip (2.2); instead we have solved the following problem, resembling Problem 1.

PROBLEM 2. *Find the constants* $\alpha, \delta, F$ *and the function* $W(\chi)$ *which is analytic and periodic on* $\psi$ *with a period* $2\pi/\lambda_0$ *in some strip*

$$\varphi_1 < \varphi < \varphi_2, \qquad -\infty < \psi < \infty \tag{6.1}$$

*and satisfy the following conditions:*

$$\left|\frac{\mathrm{d}W}{\mathrm{d}\chi} + \alpha\right|^2 = \frac{\delta^2}{\beta - 2\,\mathrm{Im}\,W/F^2} \quad (\psi = 1,\ \varphi_1 < \varphi < \varphi_2),$$

$$\mathrm{Im}\,W = 0 \qquad (\psi = 0,\ \varphi_1 < \varphi < \varphi_2).$$

Repeating all the reasonings of § 3, we conclude that the integration of system (3.5) for rational number $\lambda_0/\pi$ gives the solution of Problem 2. Having solved this system for $\lambda_0 = \pi/3$ we obtained the solution of Problem 2 with $\varphi_1 = -\varphi_0(b), \varphi_2 = \varphi_0(b)$. Solving (3.5) for other $\lambda_0$ we can obtain other solutions of Problem 2, which will describe the heavy fluid flows with a partially free boundary and partially flat bottom.

Let us denote the solution of Problem 2 by $W^{(0)}(\chi)$ to distinguish it from the exact solution $W(\chi)$ of Problem 1. The equality

$$W(\chi) \simeq W^{(0)}(\chi) \tag{6.2}$$

is fulfilled for long waves of small amplitude when the approximation of cnoidal waves is valid ($\lambda_0 \simeq 0$). We have showed also that this equality is fulfilled with good accuracy up to the highest possible amplitude ($\lambda_0 = \pi/3$). Thus one can expect the fulfilment of (6.2) for intermediate values of $\lambda_0 \in (0, \pi/3)$. The solution of Problem 2 for this $\lambda_0$ should apparently describe flows which are close to gravity waves of non-maximal amplitude.

Function $W$ is analytic and periodic along the horizontal strip (2.2). Conversely, the function $W^{(0)}$ is analytic and periodic along the vertical strip (6.1). Consequently, both of these functions are very distinct and the fulfilment of the approximate equality (6.2) cannot be expected everywhere in the $\chi$-plane. The proximity of the functions should be expected at the intersection of the strips, where both of these functions are known to be analytic. This region is rectangle $BDD'B'$ shown on figure 1($b$). If this rectangle includes one wavelength, then the function $W^{(0)}$ may be used for an approximate description of gravity waves.

The approximate equality (6.2) leads to the assumption that for the gravity wave problem there exists a small parameter $\varepsilon$ characterizing the inclination of $W$ from $W^{(0)}$ and the solution may be sought as the following series:

$$W(\chi) = W^{(0)}(\chi) + \varepsilon W^{(1)}(\chi) + \varepsilon^2 W^{(2)}(\chi) + \dots . \tag{6.3}$$

Substituting the series (6.3) into the boundary conditions (2.3), (2.4) and collecting the terms with the same degrees of $\varepsilon$, we obtain recurrent linear boundary-value problems for determining $W^{(1)}, W^{(2)}, \cdots$. Analysis shows that all $W^{(j)}$ are quasi-periodic functions of $\psi$ involving the incommensurable frequencies

$$\lambda_0, \lambda_1, \lambda_2, \dots \quad (\lambda_{j+1} > \lambda_j), \tag{6.4}$$

determined by the equation analogous to (4.2). The $\varphi$-periodic solution of Problem 1, the function $W(\varphi + \mathrm{i}\psi)$, is a non-periodic function on $\psi$, but is quasi-periodic with an infinite set of frequencies (6.4). A suitable form for this solution involving all harmonics $\lambda_j$ and generalizing the series (3.4) is

$$W(\chi) = \sum_{j_0=0}^{\infty} \sum_{j_1=0}^{\infty} \sum_{j_2=0}^{\infty} \cdots E_{j_0 j_1 j_2 \cdots} \sinh[(j_0\lambda_0 + j_1\lambda_1 + j_2\lambda_2 + \dots)\chi]$$
$$\left(E_{000\cdots} = 0, \qquad \operatorname{Im} E_{j_0 j_1 j_2 \cdots} = 0\right).$$

The approximate equality (6.2) implies that the least frequency $\lambda_0$ plays the dominant role in this series, while the rest give small additions. However, these small additions become essential if we want to study other solution branches. For the description of water waves with more than one trough and crest per period account must be taken of higher-order harmonics $\lambda_j\,(j > 0)$.

Why does the case $\lambda_0 = \pi/3$, when the system (3.5) consists of only three equations, correspond to peak amplitude waves? There is no comprehensive answer to this question. Notice that the number 3 and its multiples frequently appear in the wave problem. For example, in the Longuet-Higgins (1973) theory of the highest wave the conformal mapping of 6 successive waves into a hexagon was used. The appearance of this number is connected with the internal symmetry of the problem. An angle of 120° at the top of singular point on the free surface divides the full angle of 360° exactly into three parts. Therefore the triple turning of the wave about a singular point gives the initial configuration.

For $\lambda_0 < \pi/3$, the system (3.5) consists of $n\,(n > 3)$ equations and its integration is a more difficult problem. Owing to the integral

$$(P_1 - P_2)^2 + (P_2 - P_3)^2 + \ldots + (P_{n-1} - P_n)^2 + (P_n - P_1)^2 = \text{const.}$$

the system (3.5) for $n = 4$ may be integrated (Karabut 1995). It is expected, that a solution may be found also for other small values of $n$. However, in the general case we do not know the solution of (3.5). It would be very useful to study the properties of this system with an arbitrary number of equations.

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